A NOTE ON TRANSFORMED DENSITY REJECTION

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Abstract. In this paper we describe a version of transformed density rejection that requires less uniform random numbers. Random variates below the squeeze are generated by inversion. For the expensive part between squeeze and density an algorithm that uses a coverering with triangles is introduced.

1. Introduction

Transformed density rejection, introduced in Devroye (1986) and under a different name in Gilks and Wild (1992) and generalized in Hörmann (1995), is one of the most efficient universal methods for generating non-uniform random variates. This acceptance/rejection technique is based on the idea that the probability density function f is transformed by a strictly monotonically increasing differentiable transformation T with domain $(0, \infty)$, such that $T(f(x))$ is concave. We then say that f is *T-concave*; log-concave densities are an example with $T(x) = \log(x)$. By the concavity of $T(f(x))$ it is easy to construct a majorizing function for the transformed density as the minimum of n tangents. Transforming this function back into the origninal scale be get a *hat* function $h(x)$ for the density f. By using secants between the touching points of the tangents of the transformed density we can construct a simple lower bound $s(x)$ for the density, called *squeeze*, to reduce the number of (expensive) density evaluations. See Hörmann (1995) for details. Figure 1 illustrates the situation for the standard normal distribution and the transformation $T(x) = \log(x)$. The left hand side shows the transformed density with three tangents. The right hand side shows the density function with the resulting hat. Squeezes are drawn as dashed lines.

Obviously T must have the property that the area $\int h(x)\,dx$ below the hat is finite, and that generation from the hat distribution is easy (and fast). In the following we only consider the family T_c of transformations, where $T_0(x) = \log(x)$ and $T_c(x) = (x^c - 1)/c$, as the most important transformations. For densities with unbounded domain we must have $c \in (-1, 0]$ (Hörmann 1995), but for special cases $c > 0$ are possible (Evans and Swartz 1998).

Random variates proportional to the hat function $h(x)$ are generated by partitioning the domain of h into intervals $I_1, \cdots, I_n$ defined by the n different parts of the hat function, see figure 1. Algorithm `tdr` describes the standard version of

1991 *Mathematics Subject Classification.* Primary: 65C10 random number generation, Secondary: 65U05 Numerical methods in probability and statistics; 11K45 Pseudo-random numbers, Monte Carlo methods.

Key words and phrases. non-uniform random variates, universal method, transformed density rejection, acceptance-complement, continuous distributions, log-concave distributions, T-concave distributions.

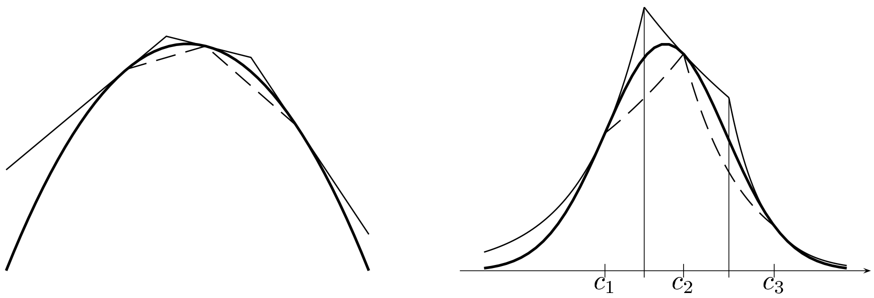

FIGURE 1. Construction of a hat function for the normal density utilizing transformed density rejection. The tangents are constructed at c_1, c_2 and c_3.

transformed density rejection. (We do not discuss here how to get suitable construction points c_i; some methods are introduced in Gilks and Wild (1992), Hörmann (1995) and Derflinger and Hörmann (1998).)

Algorithm `tdr`

Require: density $f(x)$; transformation $T(x)$, construction points $c_1, \dots, c_n$.

/* Setup */
1: Construct hat $h(x)$ and squeeze $s(x)$.
2: Compute intervals $I_1, \dots, I_n$.
3: Compute areas H_j below the hat for each interval I_j.
/* Generator */
4: **loop**
5: Generate I with probability vector proportional to $(H_1, \dots, H_n)$.
6: Generate X with density proportional to $h|_I$ (by inversion).
7: Generate $U \sim U(0,1)$.
8: **if** $U\,h(X) \le s(X)$ **then** /* evaluate squeeze */
9: **return** X.
10: **if** $U\,h(X) \le f(X)$ **then** /* evaluate density */
11: **return** X.

For step 5 *indexed search* (or *guide tables*) can be used (Chen and Asau 1974). By reusing of uniform random numbers (see (Devroye 1986, §II.3.7)) only one uniform random variate is necessary for steps 5 and 6.

The expected number of uniform random numbers per generated non-uniform random variate is therefore $2\,C = 2\int h(x)\,dx / \int f(x)\,dx \ge 2$, where C denotes the rejection constant. However concurring algorithms like the ziggurat method by Marsaglia and Tsang (1984), the table method by Ahrens (1995), or a variant of the ratio-of-uniforms method as suggested by Leydold (1999) require less than 2 uniform random numbers. These methods are acceptance/rejection techniques that uses the fact that below the squeeze immediate acceptance is possible without generating a second uniform random number.

In this paper we show that we can avoid the generation of a second random number below the squeeze and describe an efficient method for generating from the piece between probability density function and squeeze function.

2. A new approach

The given density function f can be decomposed into the discrete mixture of the squeeze and its complement, i.e.,

$$f(x) = s(x) + (f(x) - s(x)) \tag{1}$$

Generating from $s(x)$ is easy and can be done by the same method as in algrithm `tdr` for generating from the hat function via inversion. But in opposite to the standard algorithm, the domain of f is partitioned into intervals defined by the different parts of the squeeze s. Since we have a "region of immediate acceptance" below the squeeze, only one uniform random number is necessary.

Generating from a variate with density proportional to $f(x)-s(x)$ is the difficult (and expensive) part (Hörmann 1999). Let c_j, $j = 1, \dots, n$, denote the touching points of the tangents, and let $I_j = (c_j, c_{j+1}]$, $j = 1, \dots, n-1$, be the intervals between these points. Moreover let $I_0 = (x_0, c_1]$ and $I_n = (c_n, x_1)$, where (x_0, x_1) is the (not necessary bounded) domain of f.

In what follows we assume that $T(x)$ is concave, i.e., $c \le 1$ for the family T_c. In each interval $I_j = (c_j, c_{j+1}]$, the set of points $\mathcal{R}_j = \{(x,y)\colon c_j \le x \le c_{j+1}, s(x) < y \le h(x)\}$ is bounded by three curves: the squeeze and two different parts of the hat; see figure 2. By the concavity of T, the pieces of the hat function and of the squeeze are convex functions. Hence tangents on the squeeze are lower bounds;

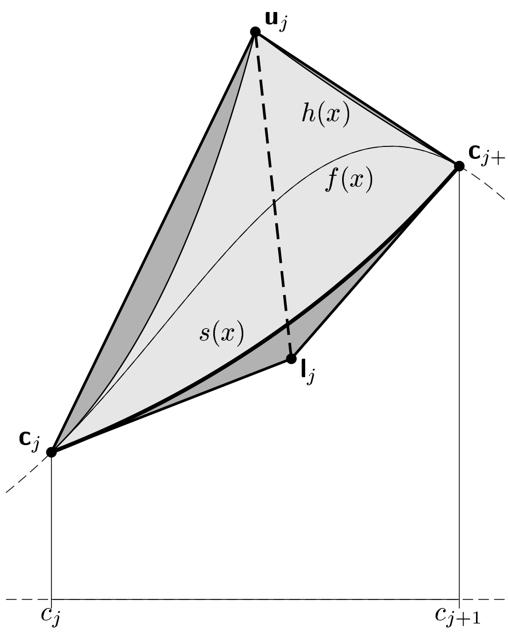

FIGURE 2. The area between squeeze $s(x)$ and hat $h(x)$ can be covered by a quadrangle $\mathcal{Q}_j$. To generate tuples uniformly distributed in $\mathcal{Q}_j$ it can be split into two triangles along diagonal $\mathbf{u}_j\mathbf{l}_j$.

secants between between two points in the same part of the hat give upper bounds. Let $\mathbf{c}_j = (c_j, f(c_j))$, let $\mathbf{l}_j$ denote the intersection of the tangents on the squeeze in the boundary points and let $\mathbf{u}_j$ be the "vertex" of graph of the hat function. Then the quadrangle $\mathcal{Q}_j$ with vertices $\mathbf{c}_j$, $\mathbf{l}_j$, $\mathbf{c}_{j+1}$ and $\mathbf{u}_j$ covers the region between hat and squeeze and consequently the region $\mathcal{R}_j$ between density and squeeze, see figure 2.

It is easy to generate a point (X, Y) uniformly distributed in $\mathcal{Q}_j$ by dividing the quadrangle into two triangles along the diagonal $\mathbf{u}_j\mathbf{l}_j$ (dashed line in figure 2). In each triangle the simple algorithm from Devroye (1986, p.570) can be used. Notice that we cannot assume that $\mathcal{Q}_j$ is convex, i.e., the diagonal $\mathbf{c}_j\mathbf{c}_{j+1}$ might not be in the quadrangle. To compile a suitable algorithm we have to be aware of the follwing two facts.

(i) There are no squeezes in the "boundary" intervals I_0 and I_n. Hence in this region we have to generate by rejection from hat distribution.

(ii) Using a partitioning of the domain of $h(x)$ into the intervals I_j together with decomposition (1) and the procedure described above results in the following algorithm: *(1)* choose an interval at random; *(2)* choose "squeeze or quadrangle" at random; *(3)* either generate from a distribution with density proportional to the squeeze and return the random variate; or *(4)* generate random tuples (X, Y) uniformly distributed in quadrangle $\mathcal{Q}_j$ until $(X, Y) \in \mathcal{R}_j$ and return X. However this approach requires the knowledge of the area of $\mathcal{R}_j$, i.e., $\int_{c_j}^{c_{j+1}} (f(x) - s(x))\, dx$. (It is not necessary for algorithm `tdr` that $\int f(x)\, dx$ is known; the normalization constants for the p.d.f. can be omitted.) This problem can be avoided by changing the procedure in the following way:

Algorithm `tdrimpr`

Require: density $f(x)$; transformation $T(x)$, construction points $c_1, \dots, c_n$.

```
/* Setup */
Construct hat h(x) and squeeze s(x).
Compute intervals I_0, ..., I_n.
Compute c_j, u_j and l_j for j = 1, ..., n-1.
Compute areas H^s_j below squeeze for j = 1, ..., n-1.
Compute areas H^l_j and H^r_j of left and right triangle, resp., inside Q_j.
Compute areas H_0 and H_n below hat for intervals I_0 and I_n.
/* Generator */
loop
   Generate I with probability vector proportional to
   (H_0, H^s_1 + H^l_1 + H^r_1, ..., H^s_{n-1} + H^l_{n-1} + H^r_{n-1}, H_n).
   if 0 < I < n then /* inner interval */
      Generate W with probability vector proportional to (H^s_I, H^l_I, H^r_I).
      if W = 1 then /* region of immediate acceptance */
         Generate X with density proportional to s|_I.
         return X.
      else /* one of the two triangles */
         Generate (X, Y) uniformly distributed in quadrangle R_I.
         if s(X) < X <= f(X) then /* evaluate squeeze and density */
            return X.
   else /* I = 0 or I = n; boundary intervals: no squeeze, use hat */
      Generate X with density proportional to h|_I.
```

```
20:    Generate U ~ U(0,1).
21:    if U h(X) ≤ f(X) then /* evaluate density */
22:       return X.
```

By reusing of uniform random numbers only one uniform random number is necessary for steps 8, 10 and 12. The expected number of uniform random numbers per nonuniform random variate is given by

$$\frac{\int s(x)\,dx}{\int f(x)\,dx} + 2\,\frac{\int_{I_0 \cup I_n} h(x)\,dx}{\int f(x)\,dx} + 2\,\frac{\sum_{j=1}^{n-1} |\mathcal{Q}_j|}{\int f(x)\,dx} \geq 1 \tag{2}$$

Notice that for increasing number n of touching points the first part in equation (2) tends to one, where the second and third part converges to zero.

Notice that the rejection constant C' of the new algorithm is given by $C' = (\int s(x)\,dx + \sum_{j=1}^{n-1} |\mathcal{Q}_j|)/\int f(x)\,dx$, where $|\mathcal{Q}_j|$ denote the area of $\mathcal{Q}_j$. We immediately find $C' \geq C$ with $(C' - C)/C = O(n^4)$. $C' - C$ is indicated by the dark shaded region in figure 2. Notice that $C - 1 = O(n^2)$ (Leydold and Hörmann 1998). Thus the increase of required loops by using the new approach is rather small.

3. Possible Variants

The new procedure works analogously for convex $T(x)$ ($c \geq 1$ for the family T_c), and/or densities f where $T(f(x))$ is convex (see Evans and Swartz (1998)).

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