Paired Comparison Preference Models

The prefmod Package: Part I

Some Examples

Regina Dittrich & Reinhold Hatzinger

Department of Statistics and Mathematics, WU Vienna

Paired Comparison Models

General Considerations:

the paired comparison models we consider in this talk are based on the Bradley-Terry Model (Biometrika, 1952)

- and will be used as statistical models
  describing and explaining data
- the models are similar to regression models (ANOVA)
  other types of dependent variables (multivariate, nominal)
  Generalised Linear Models
- but will **not** be considered as measurement models
  in this case one should use another approach
  e.g. taken by Winkelmaier in the **R** package `eba`

Paired Comparisons

- method of data collection
- given a set of J items

- individuals are asked to judge pairs of objects

j preferred to k

k preferred to j

- aim is to rank objects into a preference order
- obtain an overall ranking of the objects

Overview

- LLBT models: loglinear Bradly-Terry models

- Basic LLBT

- Extended LLBT
  undecided response
  subject covariates
  object specific covariates

- Pattern Models
  Paired comparison −▷ pattern models
  Ranking −▷ pattern models
  Rating −▷ pattern models

The Basic Bradley-Terry Model (BT)

for the each comparison (jk) of object j to object k we observe:

- $n_{(j \succ k)}$ ... the number of times j is preferred to k
- $n_{(k \succ j)}$ ... the number of times k is preferred to j

$$\boxed{N_{(jk)} = n_{(j \succ k)} + n_{(k \succ j)}}$$

total number of responses to comparison (jk)

the probability that j is preferred to k in comparison (jk)

$$\boxed{P(j \succ k) = \frac{\pi_j}{\pi_j + \pi_k}}$$

π's are a called *worth parameters* and are non-negative numbers describing the location of the objects

The Basic Loglinear BT Model (LLBT)

the model can be formulated as a log-linear model following the usual Multinomial / Poisson - equivalence.

the expected value $m_{(j \succ k)}$ of $n_{(j \succ k)}$ is $\boxed{m_{(j \succ k)} = N_{(jk)} p_{(j \succ k)}}$

$$\boxed{P(j \succ k) = \frac{\pi_j}{\pi_j + \pi_k} = c_{(jk)} \frac{\sqrt{\pi_j}}{\sqrt{\pi_k}}}$$

where $c_{(jk)}$ is constant for a given comparison

then our basic paired comparison model for one comparison is

$$\boxed{\ln m_{(j \succ k)} = \mu_{(jk)} + \lambda_j - \lambda_k}$$

λ's are the object parameters
μ's are nuisance parameters

this model formulation is feasible for further extensions

terms and relations

- relation between π and λ:

$$\boxed{\lambda_j = \ln \sqrt{\pi_j}}$$

$$\boxed{\pi_j = \exp 2\lambda_j}$$

- identifiability of πs is obtained by the restriction $\pi_J = 1$ via $\lambda_J = 0$
- the worth parameters are calculated by

$$\boxed{\pi_j = \frac{\exp(2\lambda_j)}{\sum_j \exp(2\lambda_j)}, \; j = 1, 2, \ldots, J}$$

where $\sum_j \pi_j = 1$

LLBT Design Structure

- for 3 objects we have 3 comparisons

PC	*decision*	*counts*	μ	λ_1	λ_2	λ_3
(12)	O_1	$n_{(1 \succ 2)}$	1	1	-1	0
(12)	O_2	$n_{(2 \succ 1)}$	1	-1	1	0
(13)	O_1	$n_{(1 \succ 3)}$	2	1	0	-1
(13)	O_3	$n_{(3 \succ 1)}$	2	-1	0	1
(23)	O_2	$n_{(2 \succ 3)}$	3	0	1	-1
(23)	O_3	$n_{(3 \succ 2)}$	3	0	-1	1

the design structure consists of counts (dependent variable) and the design matrix **X** with:
μ which is a factor (dummies for μ_1, μ_2, μ_3) and
variates for the objects O_1, O_2, O_3

LLBT Model Structure

model formula for the first line is:

$$\ln m_{(1 \succ 2)} = \mu_1 + \lambda_1 - \lambda_2$$

general matrix notation:

$$\begin{pmatrix} \ln m_{(1 \succ 2)} \\ \ln m_{(2 \succ 1)} \\ \ln m_{(1 \succ 3)} \\ \ln m_{(3 \succ 1)} \\ \ln m_{(2 \succ 3)} \\ \ln m_{(3 \succ 2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$$

WINE example (ficticious)

4 sicilian red wines have been tasted pairwise by 1000 people (all of them prefer red to white wines)

aim of the study:

- preference order for different wines

data:

- PC-responses about the choices of 4 selected wines
- the wines are:
- O1 = Nero D'**A**vola (also called 'Calabrese')
- O2 = **G**aglioppo (a red of Calabrian origin frequently grown in Sicily
- O3 = **P**erricone (Pignatello - esoteric, robust red)
- O4 = **N**erello (Mascalese - strong red)

Data preparation

- 4 objects (items) A,G,P,N
- number of comparisons is $4 \cdot 3/2 = 6$
- for data input the ordering of comparisons is important

	wine first in comparison			
wines	A_1	G_2	P_3	N_4
A_1	–			
G_2	V1	–		
P_3	V2	V3	–	
N_4	V4	V5	V6	–

V1	V2	V3	V4	V5	V6
(12)	(13)	(23)	(14)	(24)	(34)
(A,G)	(A,P)	(G,P)	(A,N)	(G,N)	(P,N)

Coding

One possible coding for each comparison (V1, ... V6) is

$$(jk) = \begin{cases} 1 & \text{if first object is preferred to second object} \quad (j \succ k) \\ -1 & \text{if second object is preferred to first object} \quad (k \succ j) \end{cases}$$

First respondent gave the following responses:

V1	V2	V3	V4	V5	V6
(12) (A,G)	(13) (A,P)	(23) (G,P)	(14) (A,N)	(24) (G,N)	(34) (P,N)
1	1	1	1	-1	-1

NOTE:

- missing responses should be coded with `NA`
- if the coding is **not** $(1, -1)$ but any other two numbers, the smaller number means the first object is preferred e.g. a coding with $(0, 1) \Rightarrow 0$ means first object preferred

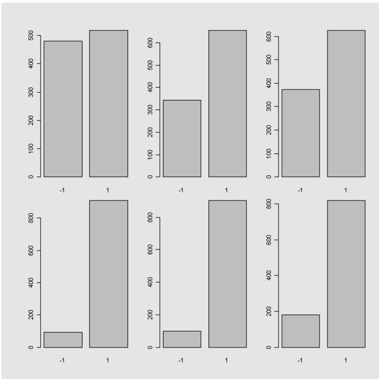
Wine data

- preparations

```
> library(prefmod)
> load("wine.Rdata")
> wnames <- c("A", "G", "P", "N")
```

produce barplots for all comparisons

```
> par(mfrow = c(2, 3))
> for (i in 1:6) barplot(table(wine[, i]))
> par(mfrow = c(1, 1))
```



Function: `llbtPC.fit()`

- preparations

```
> library(prefmod)
> load("wine.Rdata")
> wnames <- c("A", "G", "P", "N")
```

- fit simple model

```
> m1 <- llbtPC.fit(wine, nitems = 4, obj.names = wnames)
> m1
```

```
> m1
Call:

gnm(formula = formula, eliminate = elim, family = poisson, data = dfr)

Coefficients of interest:
    A      G      P      N
1.118  1.073  0.794     NA

Deviance:             2.1059
Pearson chi-squared: 2.1101
Residual df:          3
```

- model fit was done by the package `gnm()`
- We see (parameter) estimates for $\lambda_1, \lambda_2, \lambda_3$ and $\lambda_4 = NA$ (is set to zero because of restriction for estimation)
- We do not see parameter estimates for $\mu_1, \mu_2, \mu_3, \mu_4$ these are nuisance parameters and they are `eliminated` by the procedure (fitted, but not displayed)
- Does the model fit?
  We can use the Deviance 2.106
  degrees of freedom 3

```
> dev1 <- round(m1$deviance, digits = 5)
> df1 <- m1$df.residual
> prob1 <- 1 - pchisq(dev1, df1)
> print(prob1)
[1] 0.55073
```

  Model fit is OK (probability is > 0.05)

- Deviance: $\sum o_{ij} \ln (\frac{o_{ij}}{e_{ij}}) = \sum n_{ij} \ln (\frac{n_{ij}}{m_{ij}})$

  under full modell $n_{ij} = m_{ij}$

- χ^2 – statistic: $\sum \frac{(o_{ij}-e_{ij})^2}{e_{ij}} = \sum \frac{(n_{ij}-m_{ij})^2}{m_{ij}}$

Functions: `llbt.worth()`, `plotworth()`

- calculate worth parameters

```
> worth1 <- llbt.worth(m1)
```

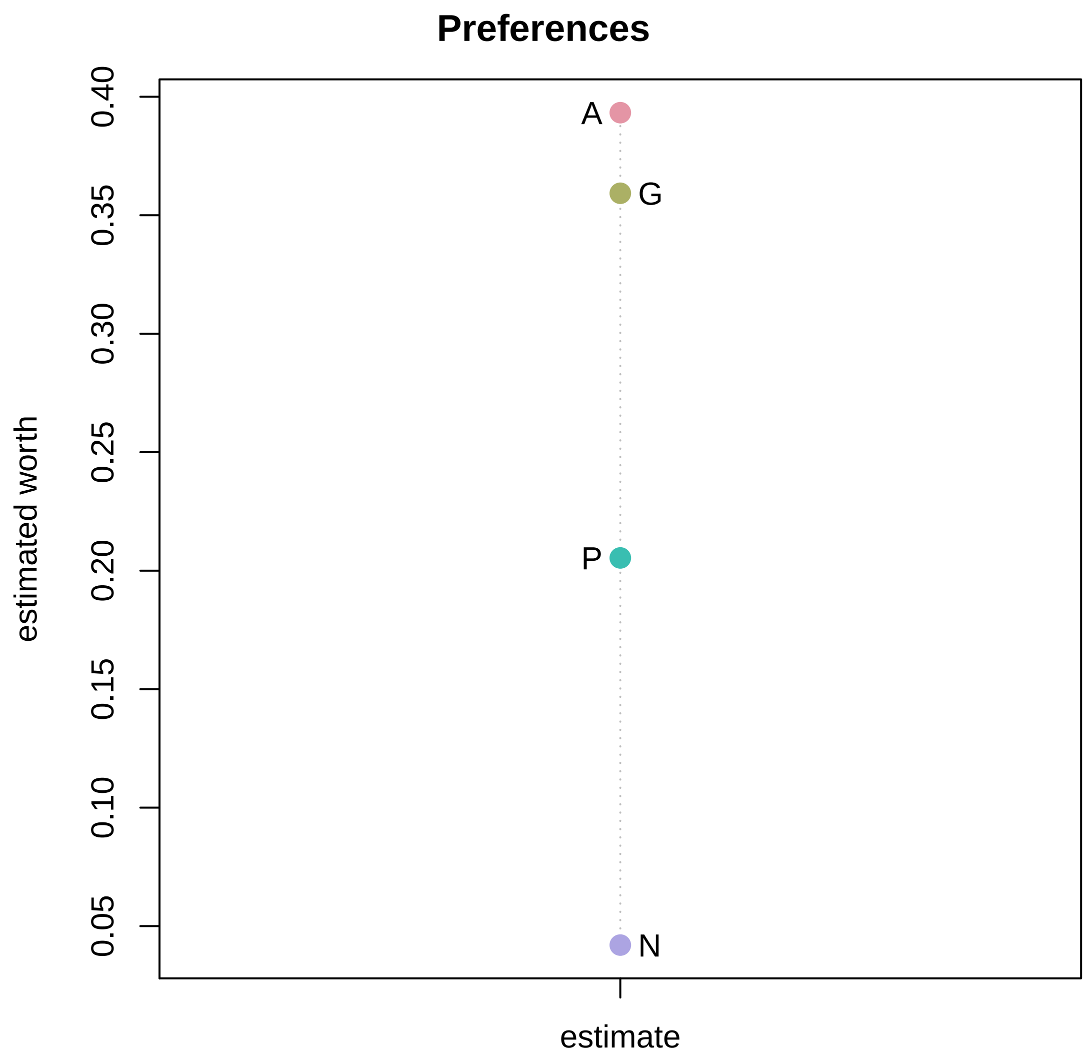
Remember: $\pi_j = \dfrac{\exp(2\lambda_j)}{\sum_j \exp(2\lambda_j)}$ $\lambda_A = 1.118,\ \lambda_G = 1.073,\ \lambda_P = 0.794,\ \lambda_N = 0$

- or get estimates

```
> est1 <- llbt.worth(m1, outmat = "lambda")
```

- plot worth or estimates

```
> plotworth(worth1, ylab = "estimated worth")
> plotworth(est1)
```


Preferences
estimated worth
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
A
G
P
N
estimate

- How to get p.e. from `m1` (which is a gnm object)

all parameter estimates $(\mu_1, \mu_2, \ldots \lambda_1, \lambda_2, \ldots)$

```
> m1$coefficients
     A      G      P      N
1.1183 1.0731 0.7935     NA
attr(,"eliminated")
[1] 6.2136 6.1628 6.1760 5.6880 5.7241 5.9282
```

- How to get parameter estimates ofInterest only?$(\lambda_1, \lambda_2, \ldots)$

```
> c1 <- coef(m1)
> c1
Coefficients of interest:
     A      G      P      N
1.1183 1.0731 0.7935     NA
```

- Replace last parameter (NA) with 0 and give names

```
> c1 <- c(c1[1:3], 0)
> names(c1) <- c("A", "G", "P", "N")
```

- How to get the μ's ?$(\mu_1, \mu_2, \ldots)$

```
> mu <- attr(m1$coefficients, "eliminated")
> mu[1:6]
[1] 6.2136 6.1628 6.1760 5.6880 5.7241 5.9282
> mu[4]
[1] 5.688
```

- How to get s.e. from `m1`

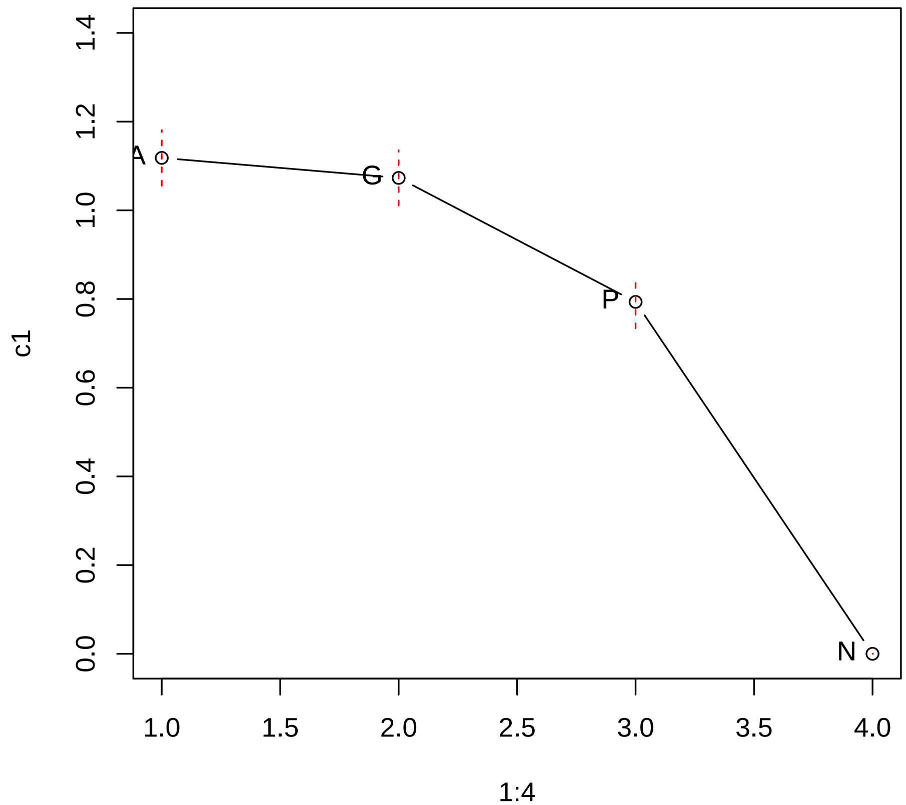
```
> cov <- vcov(m1)
> se <- sqrt(diag(cov))
```

- How to make a confidence interval plot

```
> up1 <- c1 + se * 1.96
> lo1 <- c1 - se * 1.96
```

```
> cbind(lo1, c1, up1)
      lo1     c1     up1
A 1.05548 1.1183 1.18114
G 1.01080 1.0731 1.13542
P 0.73392 0.7935 0.85308
N 0.00000 0.0000 0.00000
```

```
> plot(1:4, c1, type = "b", ylim = c(0, 1.4))
> for (i in 1:4) {
+     lines(rep(i, 2), c(lo1[i], up1[i]), col = "red", lty = "dashed")
+     text(rep(i, 2), c1[i], names(c1)[i], pos = 2)
+ }
```



Parameter Interpretation

- log odds for preferring **A** to **N**

$$\ln\left(\frac{P_{(A\succ N)}}{P_{(N\succ A)}}\right) = \ln\left(\frac{\frac{m_{(A\succ N)}}{N_{(A,N)}}}{\frac{m_{(N\succ B)}}{N_{(A,N)}}}\right) = \ln\left(\frac{m_{(A\succ N)}}{m_{(N\succ B)}}\right) = \ln m_{(A\succ N)} - \ln m_{(N\succ A)}$$

insert equations

$$\begin{aligned} \ln m_{(A\succ N)} &= \mu_{(A,N)} + \lambda_A - \lambda_N \\ -\ln m_{(N\succ A)} &= -\mu_{(A,N)} + \lambda_A - \lambda_N \end{aligned}$$

- log odds preferring **A** to **N** are $\boxed{2(\lambda_A - \lambda_N)}$ is 2.237

parameter estimates are stored in `c2[1] and c2[4]`

λ_A is 1.118 and λ_N is 0 (reference object)

- the odds for preferring **A** to **N** are $\exp(2.237) = 9.362$

- How would you calculate the expected preferences for $m_{(A \succ G)}$ and for $m_{(G \succ A)}$?

- the λ parameter estimates are:

```
> coef(m1)
Coefficients of interest:
     A      G      P      N
1.1183 1.0731 0.7935     NA
```

- the estimates for the μs are:

```
> attr(m1$coefficients, "eliminated")
[1] 6.2136 6.1628 6.1760 5.6880 5.7241 5.9282
```

- Hint 1: Use model formula to calculate the logarithm of the expected numbers with $\ln m_{(A \succ G)} = \mu_{(A,G)} + \lambda_A - \lambda_G$
- Hint 2: the expected numbers are $\exp(\ln m_{(A \succ G)})$

- The observed preferences (counts) $n_{(i \succ j)}$ are stored in

```
> m1$y
1.V11 1.V12 1.V21 1.V22 1.V31 1.V32 1.V41 1.V42 1.V51 1.V52 1.V61 1.V62
  519   481   656   344   626   374   908    92   902    98   819   181
```

- The expected preferences $m_{(i \succ j)}$ are stored in

```
> m1$fitted.values
 1.V11  1.V12  1.V21  1.V22  1.V31  1.V32  1.V41  1.V42  1.V51  1.V52
522.59 477.41 656.92 343.08 636.27 363.73 903.49  96.51 895.32 104.68
 1.V61  1.V62
830.19 169.81
```

- The raw residuals are the difference between observed and expected preferences

```
> m1$y - m1$fitted.values
    1.V11     1.V12     1.V21     1.V22     1.V31     1.V32     1.V41
 -3.58549   3.58549  -0.92408   0.92408 -10.27047  10.27047   4.50957
    1.V42     1.V51     1.V52     1.V61     1.V62
 -4.50957   6.68499  -6.68499 -11.19455  11.19455
```

Example: CEMS exchange programme

students of the WU can study abroad visiting one of currently 17 CEMS universities

aim of the study:

- preference orderings of students for different locations
- identify reasons for these preferences

data:

- PC-responses about their choices of 6 selected CEMS universities for the semester abroad (London, Paris, Milan, Barcelona, St.Gall, Stockholm)
- answer: *can not decide* was allowed
- several covariates (e.g., gender, working status, language abilities, etc.)

LLBT Extensions Overview

- ▷ undecided responses (ties)
- ▷ subject covariates
- ▷ object specific covariates

Undecided Response

- undecided   3 possible responses

for each comparison between two objects j and k the response can be:

$$j \succ k$$
$$k \succ j$$
$$j = k$$

In our CEMS–example: comparing London (LO) to Paris (PA) the response can be:

$$\text{LO} \succ \text{PA}$$
$$\text{PA} \succ \text{LO}$$
$$\text{LO} = \text{PA}$$

LLBT with Undecided Response

Using the respecification of the probabilities suggested by Davidson and Beaver (1977):

the LLBT model formulas for the comparison (jk) are now:

$$\ln m_{(j \succ k)} = \mu_{(jk)} + \lambda_j - \lambda_k$$
$$\ln m_{(k \succ j)} = \mu_{(jk)} - \lambda_j + \lambda_k$$
$$\ln m_{(j=k)} = \mu_{(jk)} \qquad\qquad + \gamma$$

where γ is the parameter for undecided response (could also be $\gamma_{(jk)}$)

λ's are the object parameters

μ's are nuisance parameters

Interpretation of parameter γ

- log odds for preferring $(j \succ k)$ to "no decision" $(j = k)$

$$\ln\left(\frac{P_{(j\succ k)}}{P_{(j=k)}}\right) = \ln m_{(j\succ k)} - \ln m_{(j=k)}$$

insert equations

$$\begin{aligned} \ln m_{(j\succ k)} &= \mu_{(jk)} + \lambda_j - \lambda_k \\ -\ln m_{(j=k)} &= -\mu_{(jk)} \qquad\qquad - \gamma \end{aligned}$$

for $\lambda_j = \lambda_k$

- log odds preferring $(j \succ k)$ to "no decision" $(j = k)$ is $\boxed{-\gamma}$

if $\gamma < 0$: there is an advantage in favour of a decision

LLBT Design Structure with undecided

PC	*decision*	*counts*	μ	γ	λ_1	λ_2	λ_3
(12)	O_1	$n_{(1\succ 2)}$	1	0	1	-1	0
(12)	no	$n_{(1=2)}$	1	1	0	0	0
(12)	O_2	$n_{(2\succ 1)}$	1	0	-1	1	0
(13)	O_1	$n_{(1\succ 3)}$	2	0	1	0	-1
(13)	no	$n_{(1=3)}$	2	1	0	0	0
(13)	O_3	$n_{(3\succ 1)}$	2	0	-1	0	1
(23)	O_2	$n_{(2\succ 3)}$	3	0	0	1	-1
(23)	no	$n_{(2=3)}$	3	1	0	0	0
(23)	O_3	$n_{(3\succ 2)}$	3	0	0	-1	1

- Design matrix consists of $\mu, \lambda_1, \lambda_2, \lambda_3$ and additionally γ where μ is a factor and the n's are the *counts*

Function: `llbtPC.fit()`

preparations

```
> library(prefmod)
> load("cpc.Rdata")
> cities <- c("LO", "PA", "MI", "SG", "BA", "ST")
```

fit simple model

```
> m2 <- llbtPC.fit(cpc, nitems = 6, obj.names = cities)
> summary(m2)
```

fit model including effect for undecided

```
> m3 <- llbtPC.fit(cpc, nitems = 6, undec = TRUE, obj.names = cities)
> summary(m3)
```

compare models

```
> anova(m3, m2)
```

Functions: `llbt.worth()`, `plotworth()`

calculate worth parameters

```
> worth2 <- llbt.worth(m2)
> worth3 <- llbt.worth(m3)
> worthmat <- cbind(worth2, worth3)
```

plot them

```
> plotworth(worthmat)
```

Subject Covariates

Are the preference orderings different for different groups of subjects?

For one subject covariate on s levels we have now

$$\ln m_{(j \succ k)|s} = \mu_{(jk)s} + \lambda_s^S + (\lambda_j^{O_j} + \lambda_{js}^{O_j S}) - (\lambda_k^{O_k} + \lambda_{ks}^{O_k S})$$

where

λ^O	object parameters (for subject baseline group)
λ^{OS}	interaction parameter between objects and subject category
λ_s^S	fixing the margin for category s of covariate S (nuisance)
μ's	nuisance parameters

LLBT Design structure with 1 subject covariate

PC	decison	counts	μ	λ^S	γ	λ_1^O	λ_2^O	λ_3^O	λ_{12}^{OS}	λ_{22}^{OS}	λ_{32}^{OS}
(12)	O_1	$n_{(1\succ 2)\|1}$	1	0	0	1	-1	0	0	0	0
(12)	no	$n_{(1=2)\|1}$	1	0	1	0	0	0	0	0	0
(12)	O_2	$n_{(2\succ 1)\|1}$	1	0	0	-1	1	0	0	0	0
⋮	⋮	⋮	⋮					⋮			
(12)	O_1	$n_{(1\succ 2)\|2}$	4	1	0	1	-1	0	1	-1	0
(12)	no	$n_{(1=2)\|2}$	4	1	1	0	0	0	0	0	0
(12)	O_2	$n_{(2\succ 1)\|2}$	4	1	0	-1	1	0	-1	1	0
⋮	⋮	⋮	⋮					⋮			

- O and OS are parameter of interest

- $MU * S$ not related to objects – nuisance parameters

Options for `llbtPC.fit()`: `formel`, `elim`

fit null model (without subject covariates)

```
> mb0 <- llbtPC.fit(cpc, nitems = 6, undec = TRUE, formel = ~1,
+     elim = ~SEX, obj.names = cities)
```

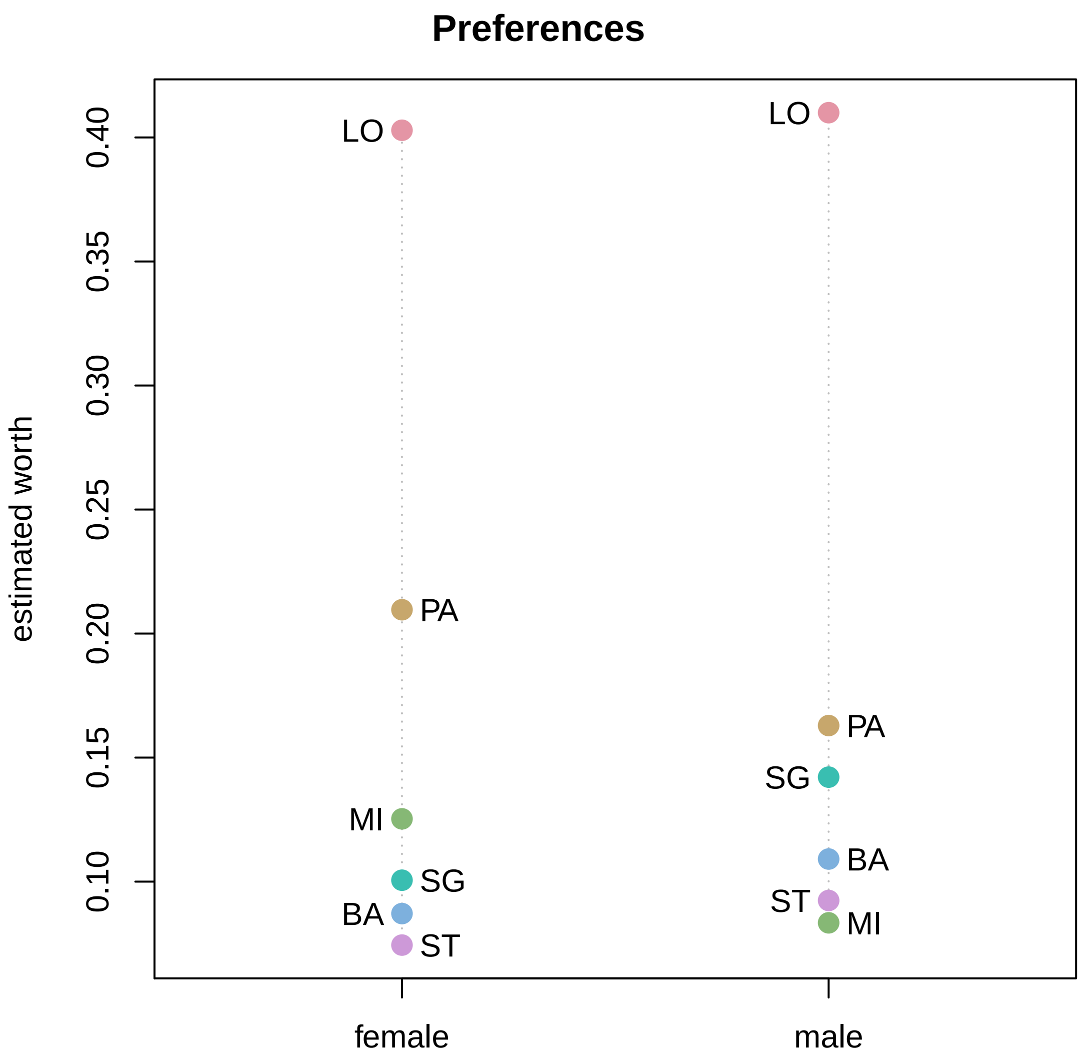
fit model with different preference scales for `SEX`

```
> mbsex <- llbtPC.fit(cpc, nitems = 6, undec = TRUE, formel = ~SEX,
+     elim = ~SEX, obj.names = cities)
```

Plot for different preference scales for `SEX`

`female=1 and male =2`

```
> wbsex <- llbt.worth(mbsex)
> colnames(wbsex) <- c("female", "male")
> plotworth(wbsex, ylab = "estimated worth")
```


Preferences
estimated worth
0.10
0.15
0.20
0.25
0.30
0.35
0.40
LO
PA
MI
SG
BA
ST
LO
PA
SG
BA
ST
MI
female
male

Interpretation of parameters ofInterest

- $\lambda_j^{O_j}$ parameter estimate for O_j for the reference group (all subjects covariates on level = 0 in dummy coding)
- $\lambda_{js}^{O_jS}$ change of $\lambda_j^{O_j}$ for group s

model: SEX	estimates for object PA		
reference group	SEX1	λ^{PA}	
	SEX2	λ^{PA}	$+\lambda^{PA:SEX2}$

Interpretation of parameters ofInterest

```
> cmsw <- coef(mbsex)
> cmsw
Coefficients of interest:
        LO         PA         MI         SG         BA         ST
 0.8444125  0.5177037  0.2604805  0.1505150  0.0788050         NA
         u     LO:SEX2    PA:SEX2    MI:SEX2    SG:SEX2    BA:SEX2
-1.3203958 -0.0994717 -0.2341781 -0.3117667  0.0647604  0.0042943
   ST:SEX2
        NA
```

model: SEX	estimates for object PA		model in **mbsex**
	λ^{PA}	$+\lambda^{PA:SEX2}$	
SEX1	0.518		$\lambda^{PA}_{female} = 0.518$
SEX2	0.518	-0.234	$\lambda^{PA}_{male} = 0.284$